The Calculus Bible: The New Testament

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Part I
Advanced Techniques of Integration

1 Trigonometric Integrals

1.1 Reduction formulas (n is an integer ≥ 2)

- $\int sin^n(u)du = \frac{-1}{n}cos(u)sin^{n-1}(u) + \frac{n-1}{n}\int sin^{n-2}(u)du$
- $\int cos^n(u)du = \frac{1}{n}sin(u)cos^{n-1}(u) + \frac{n-1}{n}\int cos^{n-2}(u)du$
- $\int sec^n(u)du = \frac{sec^{n-2}(u)}{n-1} + \frac{n-2}{n-1}\int sec^{n-2}(u)du$
- $\int tan^n(u)du = \frac{sec^{n-1}(u)}{n-1} - \int tan^{n-2}(u)du$

1.2 Power formulas (m and n are integers ≥ 2)

1.2.1 Powers of sine and cosine $\int [sin^m(x)cos^n(x)]\, dx$

- If n is odd, let $u = sin(x)$
- If m is odd, let $u = cos(x)$
- If m and n are both even, use the identities below, and then apply the appropriate reduction formula

$$sin^2(x) = \frac{1 - cos(2x)}{2}$$

$$cos^2(x) = \frac{1 + cos(2x)}{2}$$

1.2.2 Powers of sec and tan $\int [tan^m(x)sec^n(x)]\, dx$

- If n is even, let $u = tan(x)$
- If m is odd, let $u = sec(x)$
- If m is even and n is odd, reduce to a power of $sec(x)$ and then use the sec reduction formula

1.2.3 Simplification formulas

- $\int [sin(mx)cos(nx)]\, dx = \frac{1}{2} \int [sin(m–n)x + sin(m+n)x]dx]$
- $\int [sin(mx)sin(nx)]\, dx = \frac{1}{2} \int [cos(m–n)x – cos(m+n)x]dx]$, $m > n$
- $\int [cos(mx)cos(nx)]\, dx = \frac{1}{2} \int [cos(m–n)x + cos(m+n)x]dx]$, $m > n$

1.2.4 Integrals to remember

- $\int sec(u)du = ln\,|sec(u) + tan(u)| + C$
- $\int tan(u)du = –ln\,|cos(u)| + C$

2 Trigonometric subsitutions

2.1 Used for integrals that cannot be solved with a u-substitution

- If an integral contains $\sqrt{a^2 – x^2}$, let $x = asin(\theta)$ and $dx = acos(\theta)d\theta$
- If an integral contains $\sqrt{x^2 + a^2}$, let $x = atan(\theta)$ and $dx = asec^2(\theta)d\theta$
- If an integral contains $\sqrt{x^2 – a^2}$, let $x = asec(\theta)$ and $dx = asec(\theta)tan(\theta)d\theta$

Solve the integral using reduction formulas where necessary, and then make the appropriate substitutions to convert back to x.

3 Partial Fractions

Used for $\int \frac{p(x)}{q(x)}dx$

1. If $p(x)$ is a higher order polynomial than $q(x)$, then divide first.
2. Factor the denominator.
3. For every linear factor $(ax + b)^m$, introduce m terms
$$\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + ... + \frac{R}{(ax+b)^m}$$
4. Solve for all constants.
5. Integrate each partial fraction separately.

4 Advanced integration by parts

- The integral $\int f'(x)g(x)dx$, which results from simple integration by parts, may also have to be integrated by parts.
- If the integral resulting from the second integration by parts is identical to the original integral (e.g. $\int f(x)g'(x)dx = h(x) - \int f(x)g'(x)dx$, then simplify the equation to $2\int f(x)g'(x)dx = h(x)$, and finally $\int f(x)g'(x)dx = \frac{1}{2}h(x)$

5 Improper integrals

1. If $f(x)$ is continuous on (a,∞), then $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$, provided the limit exists.
2. If $f(x)$ is continuous on $(-\infty,b)$, then $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$, provided the limit exists.
3. If $f(x)$ is discontinuous at $x=c$, $a<c<b$, but is otherwise continuous on (a,b), then:
   (a) $\int_a^b f(x)dx = \lim_{x\to c^-}\int_a^c f(x)dx + \lim_{x\to c^+}\int_c^b f(x)dx$
   (b) If either limit is non-finite, then the integral is non-finite.

Part II
Further applications of the definite integral

6 Volumes of known cross-sections

1. Express the area of the cross-section as a function of x.
2. $V = \int_a^b A(x)dx$

7 Length of a plane curve

$$L = \int_a^b \sqrt{1+[f'(x)]^2}dx$$

8 Work

1. An object must move over $[a, b]$ while subject to a variable force $F(x)$ in the direction of the motion.
2. $W = \int_a^b F(x)dx$

Part III

Parametrically defined equations

9 Definition

$$x = f(t) \quad y = g(t)$$

10 Derivatives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

11 Length of the plane curve

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Part IV

Polar Functions

12 Lines

1. $rcos(\theta) = a \quad (x = a)$ vertical line
2. $rsin(\theta) = a \quad (y = b)$ vertical line
3. $Arcos(\theta) + Brsin(\theta) = C \quad (Ax + By = C)$ skewed line, A, B, $C \neq 0$
4. $\theta = a \quad (y = mx)$ line through the origin, $m = tan(\theta)$
   (a) Ex. $\theta = \frac{\pi}{4}$

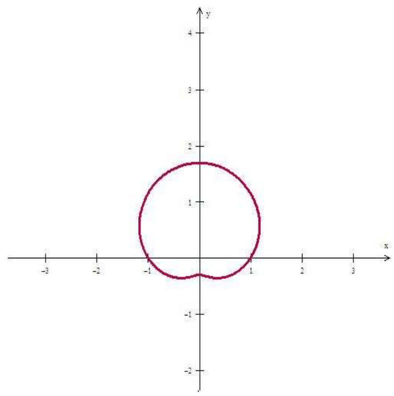
13 Circles

- $r = a$ centered at the origin, radius a
- $r = 2acos(\theta)$ centered at $(a, 0)$, radius$= a$, tangent to the y-axis
- $r = -2acos(\theta)$ centered at $(-a, 0)$, radius$= a$, tangent to the y-axis
- $r = 2asin(\theta)$ centered at $(0, a)$, radius$= a$, tangent to the x-axis
- $r = -2acos(\theta)$ centered at $(0, -a)$, radius$= a$, tangent to the x-axis

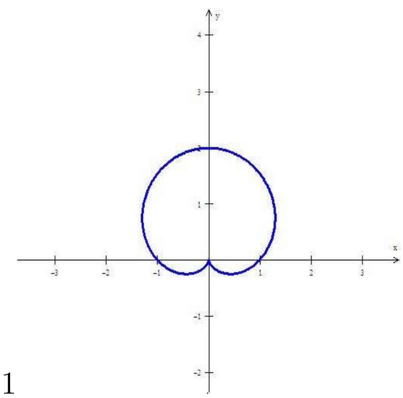
14 Rectangular hyperbolas

$$r^2 sin(2\theta) = 2a \quad (xy = a)$$

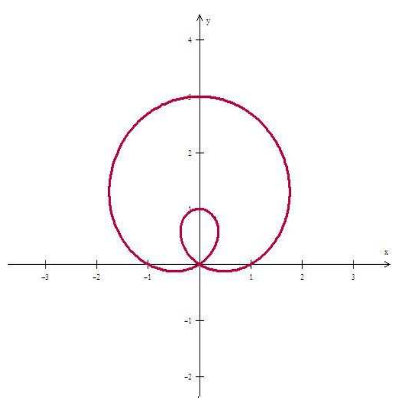
15 Limacons



- $r = a + bsin(\theta)$, $1 < \frac{a}{b} < 2$



- $r = a + bsin(\theta)$, $\frac{a}{b} = 1$

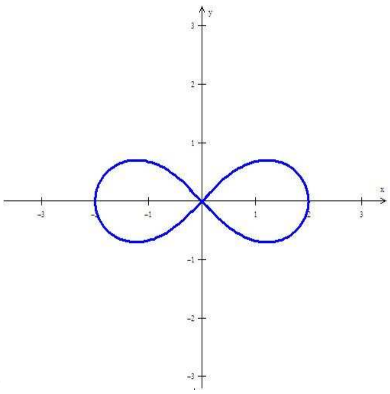


- $r = a + bsin(\theta)$, $\frac{a}{b} < 1$

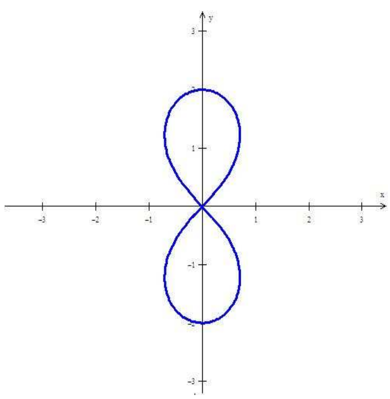
- $r = a – bsin(\theta)$ above curves reflected over x-axis
- $r = a + bcos(\theta)$ above curves rotated 90^oclockwise
- $r = a – bcos(\theta)$ above curves rotated 90^o counterclockwise

16 Lemniscates

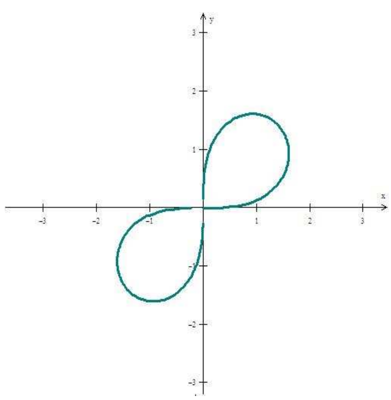
Note that a =length of one loop



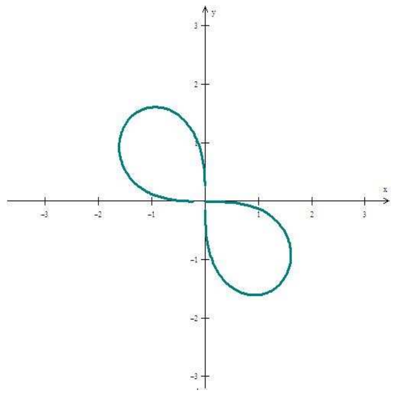
- $r^2 = a^2 cos(2\theta)$



- $r^2 = -a^2 cos(2\theta)$



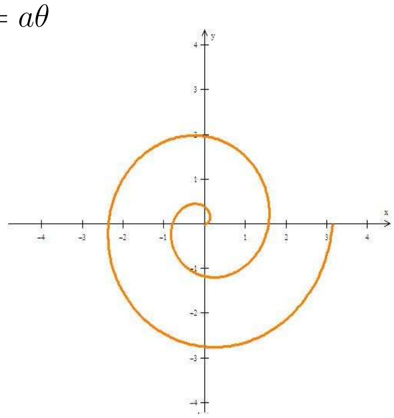
- $r^2 = a^2 sin(2\theta)$



- $r^2 = -a^2 sin(2\theta)$

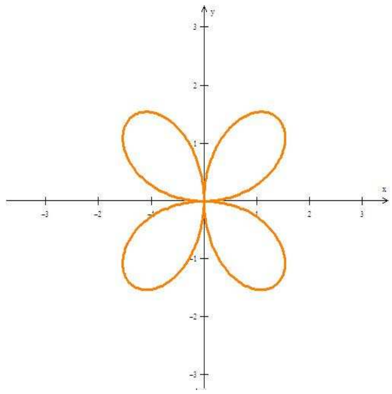
17 Spirals

$r = a\theta$

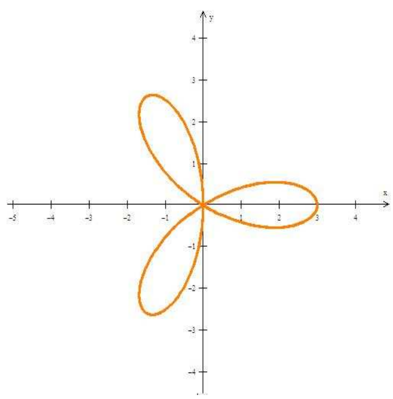


18 Roses

Note that a =length of one loop



- $r = asin(n\theta)$



- $r = acos(n\theta)$

- If n is odd, then n =number of petals
- If n is even, then $2n$ =number of petals

19 Area

- $f(\theta)$ must be continuous and non-negative
- $A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta}(r)^2 d\theta$
- Warning: in the equation of a lemniscate, r is already squared!

20 Length of the plane curve

1. Define the curve parametrically
   (a) Since $r = f(\theta)$ and $x = rcos(\theta)$, $x = f(\theta)cos(\theta)$
   (b) Since $r = f(\theta)$ and $y = rsin(\theta)$, $y = f(\theta)sin(\theta)$
2. $L = \int_{\alpha}^{\beta}\sqrt{\left[\frac{dy}{d\theta}\right]^2 + \left[\frac{dx}{d\theta}\right]^2}d\theta$

Part V

Sequences and series

21 Sequences – Convergence

1. Find the limit as $n \to \infty$ of a_n. You may use limit theorems previously proved, including L'Hopital's rule.
   (a) If the limit exists, then the sequence converges to that limit.
   (b) If the limit is non-existent, including $\pm\infty$, the sequence diverges.
2. Show that the sequence is monotone and that it is bounded in the proper direction.
   (a) Monotone defense
      i. Ratio rule
         A. $\frac{a_{n+1}}{a_n} \leq 1$ non-increasing
         B. $\frac{a_{n+1}}{a_n} \geq 1$ non-decreasing
      ii. Difference rule
         A. $a_n - a_{n+1} \geq 0$ non-increasing
         B. $a_n - a_{n+1} \leq 0$ non-decreasing
      iii. Derivative rule
         A. $f'(x) < 0$ decreasing
         B. $f'(x) > 0$ increasing

(b) Convergence

i. If $\{a_n\}$is non-decreasing and is bounded above, it is convergent.

ii. If $\{a_n\}$is non-increasing and is bounded below, it is convergent.

iii. In all other cases, $\{a_n\}$ diverges.

22 Series – Convergence

A series $\sum_{k=1}^{\infty} u_k$ converges to a sum S iff the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges to a limit. The limit of the sequence of partial sums is called the sum S of the series.

Geometric series converge iff $|r| < 1$. If the series converges, then the sum $S = \frac{a}{1-r}$

22.1 Tests for convergence

22.1.1 Divergence test

If $\lim_{k\to\infty} u_k \neq 0$, then the series $\sum_{k=1}^{\infty} u_k$ diverges. Be careful! Showing that $\lim_{k\to\infty} u_k = 0$ does not prove convergence.

22.1.2 Integral test

$\sum_{k=1}^{\infty} u_k$ converges iff $\int_1^{\infty} f(x)dx$ converges (is finite), provided that $f(x)$ is continuous, positive, and decreasing on $[1, \infty)$ and $u_k = f(k)$.

22.1.3 P-series rule

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges iff $p > 1$

22.1.4 Comparison test

If $u_k \leq v_k$ for $\forall k \geq N$, and if $\sum_{k=1}^{\infty} v_k$ is known to converge, then $\sum_{k=1}^{\infty} u_k$ converges as well.

If$u_k \geq v_k$ for $\forall k \geq N$, and if $\sum_{k=1}^{\infty} v_k$ is known to diverge, then $\sum_{k=1}^{\infty} u_k$ diverges as well.

22.1.5 Ratio Test

If $\sum_{k=1}^{\infty} u_k$ is a positively valued series and $\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$, then if:

- $\rho > 1$, the series diverges.
- $\rho < 1$, the series converges.
- $\rho = 1$, no conclusion can be reached.

22.1.6 Root test

If $\sum_{k=1}^{\infty} u_k$ is a positively valued series and $\rho = \lim_{k\to\infty} \sqrt[k]{u_k}$, then if:

- $\rho > 1$, the series diverges.
- $\rho < 1$, the series converges.
- $\rho = 1$, no conclusion can be reached.

22.1.7 Limit comparison test

If $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ are positively valued series and if $\rho = \lim_{k\to\infty} \frac{u_k}{v_k}$, then if ρ is finite and$\neq 0$, then the two series either both converge or both diverge.

22.1.8 Ratio test for absolute converegence

If $\sum_{k=1}^{\infty} u_k$ is a series and $\rho = \lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right|$, then if:

- $\rho > 1$, the series diverges.
- $\rho < 1$, the series converges.
- $\rho = 1$, no conclusion can be reached.

22.1.9 Alternating series test

If $\sum_{k=1}^{\infty} u_k$ is an alternating series and

1. $u_{k+1} \leq u_k \ \ \forall k > N$
2. $\lim_{k\to\infty} u_k = 0$

then the series is said to be at least conditionally convergent.

22.2 Theorems

1. If a series converges absolutely, then it converges.
2. If, according to the "ratio test for absolute convergence" (22.1.8), a series diverges, then there is no chance it will conditionally converge.

22.2.1 Error Theorem

If $\sum_{k=1}^{\infty} u_k$ is an alternating series and n terms are used to approximate the sum, then the error $= |S - S_n| \leq u_{n+1}$

22.2.2 Function approximations using infinite series

- The function $f(x)$ is approximated in the neighborhood of zero by the series $\sum_{k=0}^{\infty} \frac{f^{(x)}(0)}{k!} x^k$. The interval of convergence are the values of x for which the series converges.
- The function $f(x)$ is approximated in the neighborhood of $(x = a)$ by the series $\sum_{k=0}^{\infty} \frac{f^{(x)}(a)}{k!}(x - a)^k$. The interval of convergence are the values of x for which the series converges.

22.3 Function approximations by infinite series to remember

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad -1 < x < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad -\infty < x < \infty$$

$$sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad -\infty < x < \infty$$

$$cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad -\infty < x < \infty$$

$$ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k+1)} \quad -1 < x \leq 1$$

22.4 Derivatives and integrals of power series

- If $\sum_{k=0}^{\infty} c_k (x-a)^k$ is a power series, then $\frac{d}{dx} \sum_{k=0}^{\infty} c_k (x-a)^k = \sum_{k=1}^{\infty} k c_k (x-a)^{k-1}$ (the lower limit only changes if the first term of the original series is a constant
- If $\sum_{k=0}^{\infty} c_k (x-a)^k$ is a power series, then $\int \sum_{k=0}^{\infty} c_k (x-a)^k = \sum_{k=0}^{\infty} \frac{c_k}{k+1} (x-a)^{k+1} + C$

22.5 Derivation of a power series from another power series

- If $\sum_{k=0}^{\infty} c_k (x-a)^k$ is a power series, then any monomial $m(x)$ or binomial of form $ax^p + b$ can be substituted in for x in the summation to serive another power series.
- The interval of convergence can be determined by substituting $m(x)$ for x in the original interval of convergence.