The Calculus Bible

www.calcbible.com

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Introduction

This document is not meant to be a textbook, but rather a review guide or a study aide for high school students who have completed the Advanced Placement AB Calculus curriculum. All other BC Calculus topics are separately outlined in *The Calculus Bible, The New Testament* by the same author.

Part I
Functions

1 Domain, Range, Asymptotes, and Fundamental Properties

1.0.1 Polynomials

- D = $\mathbb{R}$

1.0.2 Rational

$\frac{f(x)}{g(x)}$ D = $\mathbb{R}$ – {x: $g(x) = 0$}

Vertical asymptotes

- x = a, where a is an exclusion from the domain

Horizontal symptotes

- y = 0 if $g(x)$is a higher order than $f(x)$
- y = ratio of leading coefficients if $f(x)$ and $g(x)$ are of same order

1.0.3 Trigonometric

$y = sin(x)$

- D = $\mathbb{R}$
- R = $-1 \leq y \leq 1$

$y = sec(x)$

- D = $\{x : x \neq \frac{n\pi}{2}\}$where n is an **odd** integer
- R = | y | $\geq$1

$y = tan(x)$

- D = $\{x : x \neq \frac{n\pi}{2}\}$where n is an **odd** integer
- R = $\mathbb{R}$

Identities

- $cos^2(x) + sin^2(x) = 1$
- $sin(2x) = 2sin(x)cos(x)$
- $cos(2x) = cos^2(x) – sin^2(x)$
- $\tan^2(x) + 1 = sec^2(x)$

1.0.4 Inverse Trigonometric

$y = sin^{-1}(x)$

- D = $-1 \leq x \leq 1$
- R = $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

$y = sec^{-1}(x)$

- D = $|x| \geq 1$
- R = $0 \leq y \leq \frac{\pi}{2}$ or $\pi \leq y \leq \frac{3\pi}{2}$

$y = tan^{-1}(x)$

- D = $\mathbb{R}$
- R = $\frac{-\pi}{2} < y < \frac{\pi}{2}$

1.0.5 Exponential

$y = b^x$, $b > 0$

- D = $\mathbb{R}$
- R = $y > 0$
- Remember: $y = b^x \Leftrightarrow x = log_b y$

1.0.6 Logarithmic

$y = log_b x$

- D = $x > 0$
- R = $\mathbb{R}$

Properties

- $log_b(m + n) = log_b m + log_b n$
- $log_b(\frac{m}{n}) = log_b m – log_b n$
- $log_b m^r = r log_b m$
- $log_b 1 = 0$
- $log_b b^x = x$
- $b^{log_b x} = x$

2 Absolute Value

- $|f(x)|$ "flips" all points above the x-axis
- $f(|x|)$ makes the function even

3 Inverses

- Found by switching the places of x and y and solving for y
- Remember: $y = log_b x$ and $y = b^x$ are inverses, which means $y = ln(x)$ and $y = e^x$ are inverses

4 Odd and Even Functions

4.1 A function is odd if $f(-x) = -f(x)\ \forall x \in D$

- Symmetrical with the origin
- Polynomials with all odd powers (no constants, because $4 = 4x^0$)

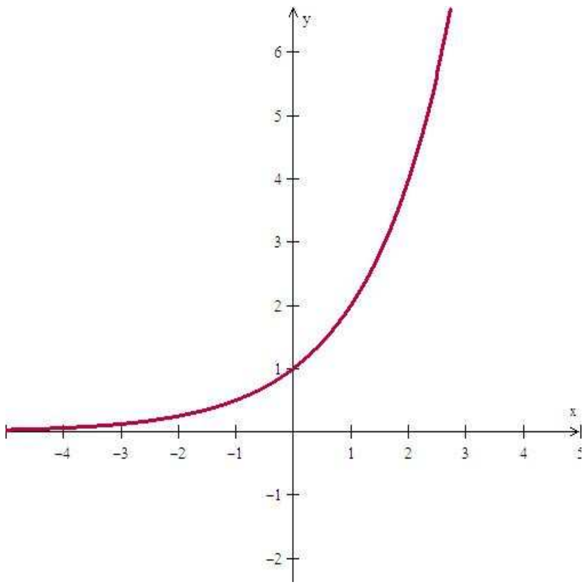
4.2 A function is even if $f(-x) = f(x)\ \forall x \in D$

- Symmetrical with the y-axis
- Polynomials with all even powers

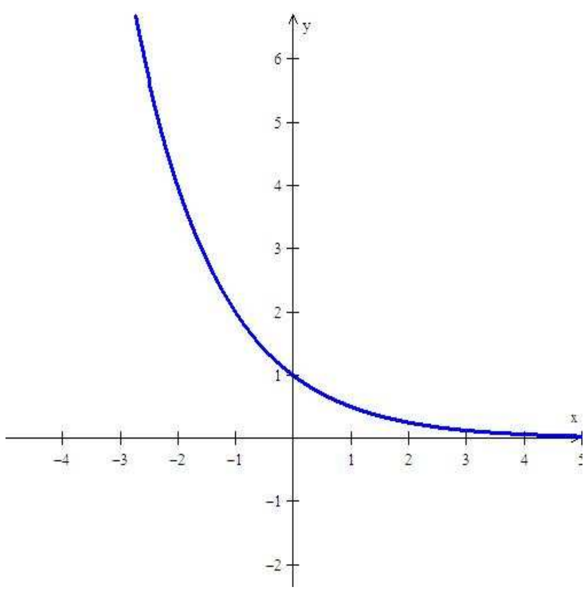
4.3 Remember, many functions are neither odd nor even

5 Graphs You Must Know

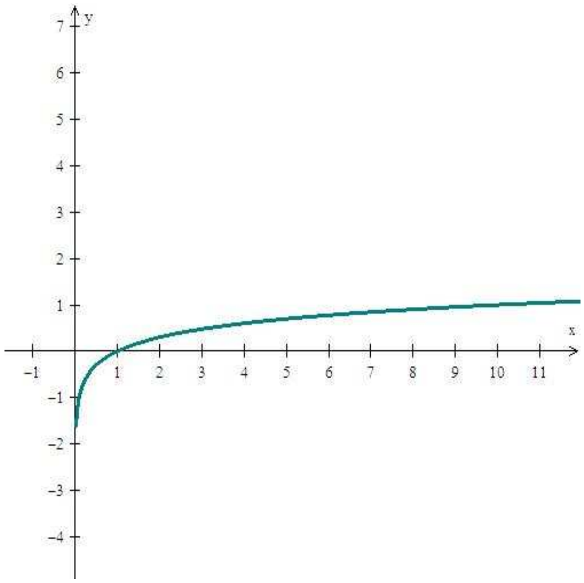
$y = b^x,\ b > 1$



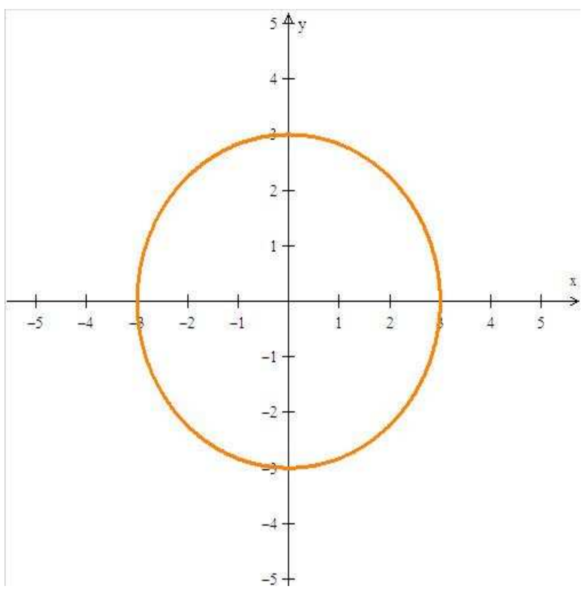
$y = b^x,\ b < 1$



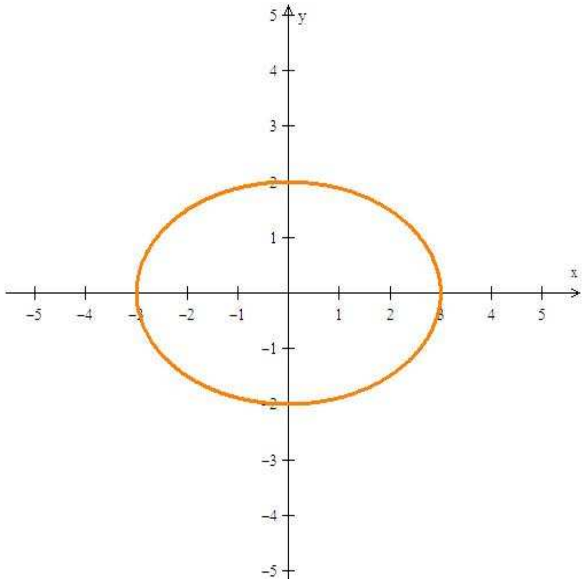
$y = log_b x \ (includes\, y = ln(x))$



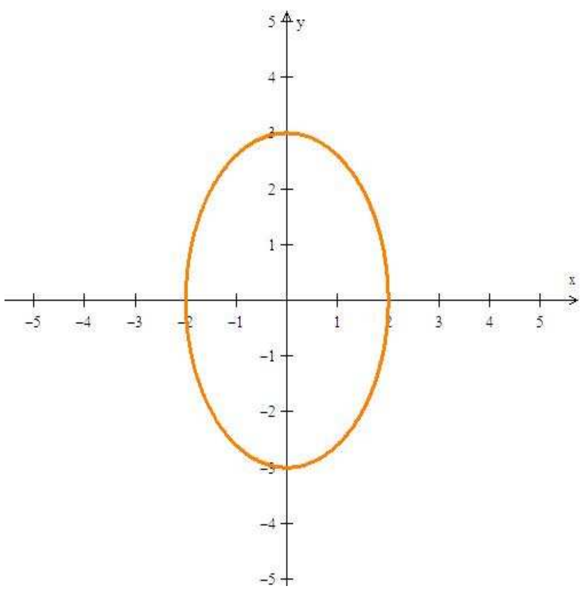
$x^2 + y^2 = r^2$**(note: r is the radius, so** $3x^2 + 3y^2 = 15$ **is a circle of radius** $\sqrt{5}$



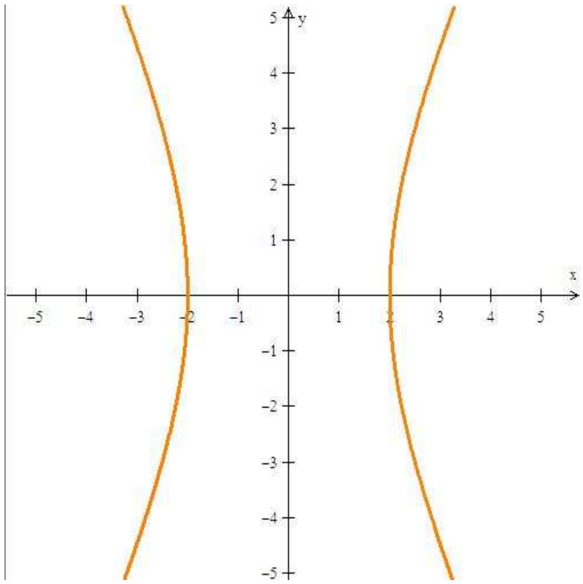
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \; a > b$



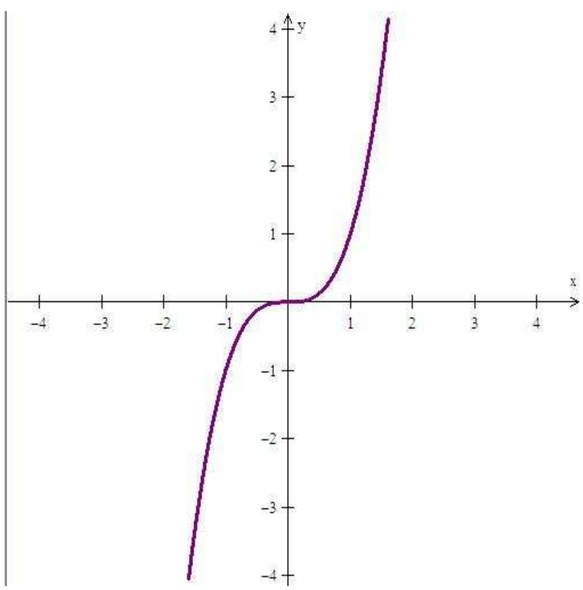
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \; a < b$



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$y = x^3$



6 Zeroes of a function

- Set the function = 0 and solve
- For cubics, find any rational root, r, and synthetically divide by $(x–r)$
  - If you cannot find a rational root, use Newton's method of approximation: $x_{n+1} = x_n – \frac{f(x_n)}{f'(x_n)}$

7 Symmetry

- y-axis: $(x, y) \rightarrow (–x, y)$
- x-axis: $(x, y) \rightarrow (x,$–y) (non-function)
- origin: $(x, y) \rightarrow (–x, –y)$
- $y = x$: $(x, y) \rightarrow (y, x)$

8 Limits

8.1 Theorems

1. Limit of a sum or difference = sum or difference of the limits
2. Limit of a product = product of the limits
3. Limit of a quotient = quotient of the limits (unless the limit of the denominator = 0)
4. Limit of the nth root = nth root of the limit

8.2 Limits you must know

1. $\lim\limits_{b \to 0^+} \frac{a}{b} = +\infty$, where $a > 0$ and finite
2. $\lim\limits_{b \to 0^-} \frac{a}{b} = –\infty$, where $a > 0$ and finite
3. $\lim\limits_{b \to 0^+} \frac{a}{b} = –\infty$, where $a < 0$ and finite
4. $\lim\limits_{b \to 0^-} \frac{a}{b} = +\infty$, where $a < 0$ and finite
5. $\lim\limits_{b \to \pm\infty} \frac{a}{b} = 0$, where a is finite
6. $\lim\limits_{x \to 0}(1 + x)^{\frac{1}{x}} = \lim\limits_{x \to \infty}(1 + \frac{1}{x})^x = e$

8.3 Non-existent limits

- $\pm\infty$us a non-existent limit ($\lim_{x\to 0} \frac{1}{x^2}$doesn't exist)
- limits which do not converge are non-existent limits
  - Ex: $\lim_{x\to 0} sin(\frac{1}{x})$ and $\lim_{x\to\infty} sin(x)$ are non-existent (fluctuates between 1 and -1)
- If $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$ then the limit does not exist
  - Ex: $\lim_{x\to 0} \frac{|x|}{x}$ does not exist

9 Continuity

9.1 Definition

A function is continuous at a if $f(a)$ exists and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

9.2 Theorems

1. If $f(x)$ and $g(x)$ are continuous at c then
   (a) $f+g$, $f-g$, $f\times g$, $f\circ g$ are continuous at $g(c)$
   (b) $\frac{f}{g}$ is continuous at c if $g(c)\neq 0$
2. If $f(x)$ is continuous on $[a,b]$, then $f(x)$ has a maximum and a minimum value on $[a,b]$
3. If $f(x)$ is continuous on $[a,b]$ and $f(a) < c < f(b)$ then $\exists$ at least one x in $[a,b]$: $f(x) = c$

Part II
Differential Calculus

10 The Derivative

10.1 Definitions (you must know them!)

1. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ (generates a slope function)
2. $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

10.2 Theorems

1. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$ where c is a constant
2. Product rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
3. Quotient rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
4. Chain rule: $\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$
5. Implicit: $\frac{d}{dx}[u(y)] = u'(y)\frac{dy}{dx}$
6. Logarithmic: If $y = f(x)^{g(x)}$, then $ln(y) = g(x)ln(f(x))$ and $\frac{dy}{dx} = y\frac{d}{dx}g(x)ln(f(x)) = f(x)^{g(x)}\frac{d}{dx}g(x)ln(f(x))$
7. Rolle's Theorem: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(b) = f(a) = 0$ then $\exists c$ in $(a, b)\ :\ f'(c) = 0$
8. Mean Value Theorem: If $f(x)$ is continuous on [a,b] and differentiable on (a,b), $\exists c$ in $(a, b)\ :\ f'(c) = \frac{f(b)-f(a)}{b-a}$

10.3 Differentiation Formulas

1. $\frac{d}{dx}[u^r] = ru^{r-1}\frac{du}{dx}$
2. $\frac{d}{dx}[sin(u)] = cos(u)\frac{du}{dx}$
3. $\frac{d}{dx}[cos(u)] = -sin(u)\frac{du}{dx}$
4. $\frac{d}{dx}[tan(u)] = sec^2(u)\frac{du}{dx}$
5. $\frac{d}{dx}[cot(u)] = -csc^2(u)\frac{du}{dx}$
6. $\frac{d}{dx}[sec(u)] = sec(u)tan(u)\frac{du}{dx}$
7. $\frac{d}{dx}[csc(u)] = -csc(u)cot(u)\frac{du}{dx}$
8. $\frac{d}{dx}[sin^{-1}(u)] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
9. $\frac{d}{dx}[tan^{-1}(u)] = \frac{1}{1+u^2}\frac{du}{dx}$
10. $\frac{d}{dx}[sec^{-1}(u)] = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$
11. $\frac{d}{dx}[a^u] = a^u ln(a)\frac{du}{dx}$
12. $\frac{d}{dx}[e^u] = e^u\frac{du}{dx}$
13. $\frac{d}{dx}[ln|u|] = \frac{1}{u}\frac{du}{dx}$

10.4 Differentiability vs. Continuity

- The fact that $f(x)$ is non-differentiable at c is not sufficient to conclude that $f(x)$ is discontinuous at c
  - For example, $y = |x–3|$
- For $f(x)$ to be differentiable at c, $f(x)$ must be continuous at c
- Continuity **does not** necessarily imply differentiability

11 Applications of the derivative

11.1 Slope of curve

The slope of a curve at point $(a, b) = f'(a)$. Remember that (a, b) is on both the curve and the tangent line.

11.2 Slope of normal line

The slope of the normal line at point $(a, b) = \frac{-1}{f'(a)}$

11.3 Curve Sketching

11.3.1 Increasing/Decreasing

- If $f'(x) > 0$ then $f(x)$ is increasing
- If $f'(x) < 0$ then $f(x)$ is decreasing

11.3.2 Critical Points

- If $f'(c) = 0$ then $(c, f(c))$ is a critical point
- If $f'(c)$ does not exist, then $(c, f(c))$ is a critical point

11.3.3 Relative extrema

- If $x = c$ is a critical point and $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c, then $(c, f(c))$ is a relative maximum
- If $x = c$ is a critical point and $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c, then $(c, f(c))$ is a relative minimum
- If $x = c$ is a critical point and $f''(c) < 0$ then $(c, f(c))$ is a relative maximum
- If $x = c$ is a critical point and $f''(c) > 0$ then $(c, f(c))$ is a relative minimum

11.3.4 Inflection points

- If concavity changes, ($f''(x)$ changes sign) at the point $(x_0, f(x_0))$, then $(x_0, f(x_0))$ is an inflection point
- If $f'(x_0) = 0$ then there is also a horizontal tangent at that point
- If $f'(x_0) = \pm\infty$ then there is also a vertical tangent at that point
- If $(c, f(c))$ is a critical point and $f'(x)$ does not change sign at $x = c$, then $(c, f(c))$ is an inflection point
- If $(x_0, f(x_0))$ is an inflection point, then $f''(x_0) = 0$. Note: the converse is not necessarily true!

11.3.5 Concavity

- If $f''(x) > 0$ on (a, b) then the graph of $f(x)$ is concave up on (a, b)
- If $f''(x) < 0$ on (a, b) then the graph of $f(x)$ is concave down on (a, b)

12 Max-min word problems

1. Evaluate the function at the endpoints and at all critical points.
2. The largest value will be the absolute maximum and the smallest value will be the absolute minimum.
3. Remember, continuous functions have both an absolute maximum and an absolute minimum on a **closed interval**.

13 Motion along a line

Go from $x(t) \rightarrow v(t) \rightarrow a(t)$ by differentiating.

14 Related Rates

1. Find an equation which relates the quantity with the unknown rate of change to quantitie(s) whose rate(s) of change are known.
2. Differentiate both sides of the equation implicitly with respect to time.
3. Solve for the derivative that represents the unknown rate of change.
4. **Then** evaluate this derivative at the conditions of the problem.

Part III
Integral Calculus

15 Antiderivatives

15.1 Definition

$F(x)$ is an antiderivative of $f(x)$ if and only if $F'(x) = f(x)$

15.2 Applications

15.2.1 Exponential growth and decay

- If you know that $\frac{dy}{dt} = ky$ then $y = Ce^{kt} \Leftrightarrow y = y_0e^{kt}$ where $y_0 = y(0)$
- k is the growth rate and may be given as a percent
- $T_{double} = \frac{ln2}{k}$, $T_{halve} = -\frac{ln2}{k}$

15.2.2 Differential Equations

1. Separate the variables so that the equation reads $f(y)dy = g(x)dx$.
2. Integrate both sides, adding the constant to the side with the independent variable. i.e. $\int f(y)dy = \int f(x)dx + C$
3. Solve for y if possible.

16 Techniques of Integration

16.1 Integrals you must know

1. $\int du = u + C$
2. $\int a\,du = au + C$
3. $\int u^r\,du = \frac{u^{r+1}}{r+1} + C$
4. $\int \frac{1}{u}\,du = ln|u| + C$
5. $e^u\,du = e^u + C$
6. $\int u^r\,du = \frac{u^{r+1}}{r+1} + C$
7. $\int sin(u)\,du = -cos(u) + C$
8. $\int cos(u)\,du = sin(u) + C$

9. $\int sec^2(u)\,du = tan(u) + C$

10. $\int csc^2(u)\,du = -cot(u) + C$

11. $\int sec(u)tan(u)\,du = sec(u) + C$

12. $\int csc(u)cot(u)\,du = -csc(u) + C$

13. $\int tan(u)\,du = -ln|cos(u)| + C$

14. $\int cot(u)\,du = ln|sin(u)| + C$

15. $\int \frac{1}{\sqrt{1-u^2}}\,du = sin^{-1}(u) + C$

16. $\int \frac{1}{1+u^2}\,du = tan^{-1}(u) + C$

17. $\int \frac{1}{u\sqrt{u^2-1}}\,du = sec^{-1}(u) + C$

16.2 U-substitution

The purpose of a u-substitution is to make a difficult integral look like one of the integrals above.

16.3 Inverse chain rule theorem

$$\int f(g(x))g'(x) = F(g(x)) + C$$

This theorem can be used to solve u-substitution integrals like $\int 2x\sqrt{1-9x^2}dx$.

16.4 Integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

17 The Definite Integral

17.1 Definition

$$\int_a^b f(x)\,dx = \lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} f({x_k}^n)\Delta x_k$$

17.2 Fundamental theorems

- $\int_a^b f(x)\,dx = F(b) – F(a)$
- $\frac{d}{dx}\int f(x) = f(x)$
- $\int \frac{d}{dx}f(x)dx = f(x) + C$
- $\int_a^x f(t)dt = F(x)$ where $F(a) = 0$
- $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

17.3 Approximations to the definite integral (n will be given)

17.3.1 Riemann sums using midpoints

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\left[y_1 + y_2 + ... + y_n\right]$$

17.3.2 Trapezoids

$$\int_a^b f(x)dx \approx \frac{b–a}{2n}\left[f(a) + 2f(x_1) + 2f(x_2) + ... + f(b)\right]$$

17.4 Definite integral as area

- $\int_a^b f(x)dx$ =area between $f(x)$ and the x-axis on $[a, b]$ if $f(x) \geq 0$ on $[a, b]$
- $\left|\int_a^b f(x)dx\right|$ =area between $f(x)$ and the x-axis on $[a, b]$ if $f(x) \leq 0$ on $[a, b]$
- $\int_c^d u(y)dy$ =area between $u(y)$ and the y-axis on $[c, d]$ if $u(y) \geq 0$ on $[c, d]$
- $\left|\int_c^d u(y)dy\right|$ =area between $u(y)$ and the y-axis on $[c, d]$ if $u(y) \leq 0$ on $[c, d]$

18 Applications of the Definite Integral

18.1 Motion along a line

- Go from $a(t) \to v(t) \to x(t)$ by integrating.
- Remember, the given conditions will determine the values of the constants generated by integration.
- Displacement = $\int_{t=a}^{t=b} v(t)dt$
- Distance traveled is the sum of the distances traveled right and left. Evaluate the position at time= a, at every value of time where $v(t) = 0$, and at time= b. Add up the distanced traveled on each time interval.

18.2 Mean value (average value of a function) on $[a, b]$

$$f_{av} = \frac{\int_a^b f(x)dx}{b-a}$$

18.3 Area between two curves

When finding the area between two curves, either axis can be used for orientation. **However**, the variable of integration **must** be the same as the axis of orientation.

- X-axis: $R = \int_a^b [f(x)-g(x)]\, dx$
- Y-axis: $R = \int_c^d [u(y)-v(y)]\, dy$

18.4 Volume of a solid of revolution

18.4.1 Around an axis

The axis of revolution is determined in the problem. However, either axis can be used for orientation. If the axis of orientation is different from the axis of revolution, the volume **must** be found using shells.

In general, it is simplest to use disks/washers for volumes of revolution around the x-axis, and shells for volumes of revolution around the y-axis.

- Disks with one curve (axis of revolution=axis of orientation=variable of integration):

$$x-axis:\ V = \pi \int_a^b [f(x)]^2\, dx$$

$$y-axis:\ V = \pi \int_c^d [u(y)]^2\, dy$$

- Washers with two curves (axis of revolution=axis of orientation=variable of integration)

$$x-axis:\ V = \pi \int_a^b \left[[f(x)]^2-[g(x)]^2\right] dx$$

$$y-axis:\ V = \pi \int_c^d \left[[u(y)]^2-[v(y)]^2\right] dy$$

18.4.2 Around a line

If the line is parallel to the x-axis, then your variable of integration must be x. If the line is parallel to the y-axis, then your variable of integration must be y. In either case, remember:

$$V = \pi \int_a^b \left(R^2-r^2\right) dr$$

Both radii must be in terms of the appropriate variable.

18.4.3 Volumes of known cross sections

If the cross-sections are perpendicular to the x-axis, then your variable of integration is x. If the cross-sections are perpendicular to the y-axis, then your variable of integration is y. In either case, remember:

$$V = \int_a^b A(x)dx$$

$$V = \int_c^d A(y)dy$$